Exotic Equity in Practice

Modelling Framework and Issues

Study carried out by the Quantitative Practice

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AVRIL 2020

SUMMARY

Introduction

The investment banking activity is about offering investment products to its customers, mostly institutional investors (asset managers, hedge funds, insurances...). Traditional assets, such as stocks and bonds, are no more of main interest. Indeed, investors are not satisfied with the risk-return profile they offer. This is why banks offer « alternative » solution, known as structured or exotic products. Structuration teams keep designing new products, most of them becoming more and more complex. It is a financial engineering challenge to always offering specific products to customers who have very specific needs.

From a financial modelling point of view, this activity does bring a lot of questions, both theoretical and practical. Indeed, once a bank has settled a deal with one of its customers, it carries a risky position on its portfolio. A risky position means a position which value can evolve positively or negatively depending on the market movements. Clearly, a bank cannot afford its results to be driven by non-controlled events. The first goal of financial modelling is to find models which are capable of explaining and controlling the profits and losses (P&L) of the bank. In order to reach that goal, traditional financial mathematics focuses on the volatility of the assets. Even if volatility is a core topic, some others clearly deserve to be studied as well because they have significant impacts on the P&L.

The goal of this note is to introduce these topics, their last developments and to offer some alternatives in terms of risk modelling. This paper will only focus on the equity perimeter. First, we will explain the implied volatility dynamics issue. Indeed, implied volatility is a fundamental market data that bank cannot get continuously (for practical reasons). Banks have to make assumptions about how it moves with underlyings. Then, we will focus on the forward price, and, after explaining that it contains all the drift information, see how it can be used to simulate in the best way possible. Finally, we will look at the critical topic of smoothing in finance. Because financial problems are too complicated to be solved analytically, they are often solved with numerical techniques. But those techniques do not bring « smooth » results (we will give details about the true meaning of this notion). However, it is important to keep in mind that quantitative finance does require smooth values. We will then present this problem and some techniques to overcome it.

1 Implied volatility dynamics issue

The daily routine of an exotic trader during his risk management activity can be described as follows:

- The trader plugs his implied volatility surface. Obviously, volatility cannot be plugged point by point. This is why a parametric form is used to describe the smile in most investment banks.
- From that moment on, models will be calibrated on that surface, the trader can price and hedge consistently with the market.
- When the market moves significantly, the trader recalibrates his parametric form onto the new market smile.

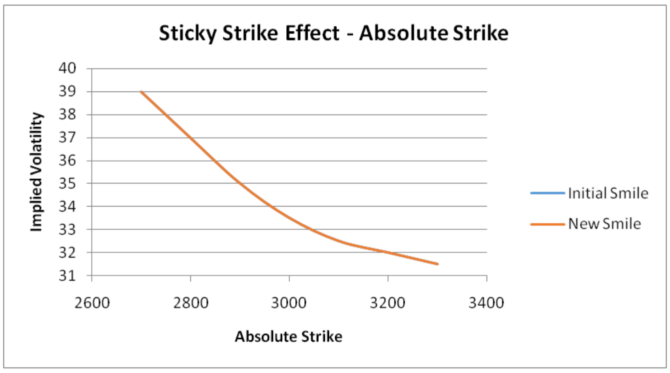
Here is the problem: because calibrating takes time and because it has to be done by a trader, it is not possible, for a bank, to have, at all time, the implied volatility of the market. Hence, between two calibrations, the trader will manage his book with an outdated implied volatility surface. But what if the spot moves a lot? It is unlikely that implied volatility will not respond to that move. Actually, there is a high negative correlation between implied volatility return and spot return. This is why banks have to make assumptions about how a spot move affects implied volatility. In common literature, one can find two extreme rules: sticky strike and sticky moneyness. We shall introduce these two concepts briefly and then we will present a new model.

1.1 Sticky Strike Rule

This is the easiest assumption one can do. When looking at the smile in absolute strike, one simply assumes that it does not change at all after a move of the spot price. Writing 0 the initial time, and 1 after the spot has moved:

$$\sigma^1(T,K) = \sigma^0(T,K)$$

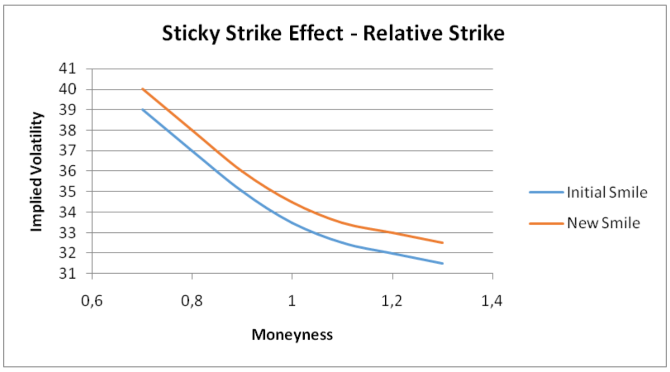
It corresponds to the following graph:



The two curves are actually the same (hence the single curve on the graph). Now, if one looks at the smile in percentage of the spot (or the forward), called moneyness, we obtain:

$$\begin{aligned}\sigma^1(T,k\%) &\overset{def}{=} \sigma^1(T,k\%xS_1)\\ &= \sigma^0(T,k\%xS_1)\\ &= \sigma^0\left(T,k\%x\frac{S_1}{S_0}S_0\right)\\ &= \sigma^0\left(T,k\frac{S_1}{S_0}\%\right)\end{aligned}$$

It means that, looking at the smile in moneyness, it does shift horizontally after the move of the spot:



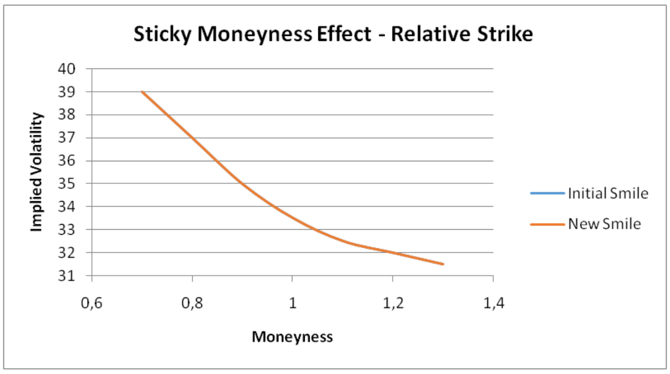
Clearly, there is no obvious reason for the smile to behave that way. It is merely a dynamics assumption.

1.2 Sticky Moneyness Rule

Whereas the previous model assumed the smile to be fixed in absolute strike, this rule assumes the smile to be fixed in moneyness. Mathematically, it comes down to the following equation:

$$\sigma^1(T, k\% x S_1) = \sigma^0(T, k\% x S_0)$$

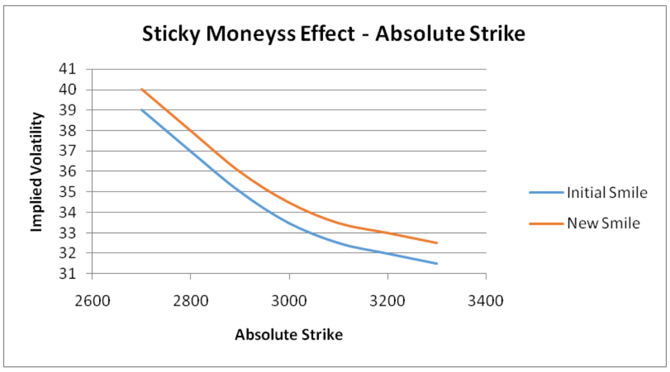
The smile expressed in moneyness thus behaves as follows:



On the contrary, looking at the smile in absolute strike leads to:

$$\sigma^1(T, K) = \sigma^1\left(T, \frac{K}{S_1}\%\right) = \sigma^0\left(T, \frac{K}{S_1}\%\right) = \sigma^0\left(T, K\frac{S_0}{S_1}\right)$$

This equation can be represented by the following graph:



Again, this dynamics rule is an assumption. Traders can assume any of those two rules to manage their portfolios. Even though in some market conditions, either sticky strike or sticky moneyness rules seems to be appropriate, it is up to the trader to decide which one to use. Besides, in reality, there is not a pure sticky strike or moneyness, but something in between. The model we suggest below does not assume any dynamics, but simply try to replicate how the smile does move with the spot.

1.3 Proposition of implied volatility dynamics

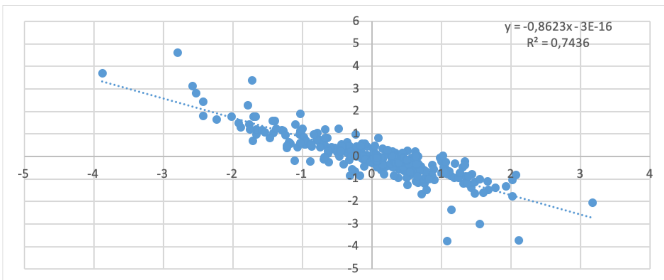
Our purpose is to estimate by how much implied volatility responds to a move of the underlying. A very basic idea is to represent it through a simple linear model:

$$d\sigma_t = \alpha + \beta dlnS_t + \epsilon$$

Implied volatility is a surface. There is, a priori, no reason for α and β to be stable through this surface. This is why one should also consider a surface for these parameters:

$$d\sigma_t^{k,T} = \sigma_{k,T} + \beta_{k,T} dlnS_t + \epsilon$$

Note that k is a relative strike. It is a percentage of the forward. Besides, when dealing with regression problems, or more generally learning problems, it performs much better when data is normalized. Indeed, the fact that data is not of the same order can bring noise to the model. Intuitively, variation of implied volatility will not be in the same order if the spot is around 1€ or around 1000 €. We have tested the above model with normalized Euro Stoxx 50 2018 data (i.e. average removed and standard deviation divided). It gives the results below.



One can see that the movements of the underlying clearly seem to explain the moves of the implied volatility. Indeed, 74.36% of the implied variation is explained by the variation of the underlying. However, is it possible to have a better model?

Linear regression is essentially a linear problem. The solution is mostly the inverse of the matrix containing the data. Computing an inverse matrix is better when the different columns are orthogonal, i.e. they give completely different information. So, if we want to improve our model, we should try to find a very orthogonal (understand independent) information of the underlying. For instance, one could add the traded volume. Indeed, it is consistent to say that the implied volatility depends on both the underlying move and the volume of trade. Here, we will focus on the structure of the volatility itself.

Indeed, implied volatility is not a single point, but an entire surface. Hence, the way it moves should depend on the shape of the surface. Imagine you have a very flat implied volatility surface. That means investors believe that whatever the spot is, implied volatility should be the same. But, if it is skewed, then one should not expect the same behavior for large and low underlying levels. This is why it could be interesting to add the skew as an explanatory variable. We decide to use the skew as information about the volatility structure. Our model thus becomes:

$$d\sigma_t^{k,T} = \alpha_{k,T} + \beta_{k,T} dlnS_t + \gamma_{k,T} Skew_t^{k,T} + \epsilon$$

Note that we could consider the skew around k. Regarding this specific example, estimating this new model using the same dataset as previously does not improve significantly the explanatory power. So, we can exclude the skew around the strike from the model (just keep in mind this feature could be useful in some cases). Maybe, one should not only consider a value around the strike, but a number that can represent the entire surface. For example, adding convexity would seem a good idea.

To sum up, the previous methodology has a huge advantage which consists in not assuming any dynamics, but building a model which is consistent with what happens. But, if we proceed with this methodology, there is absolutely no guarantee that the output surface will be arbitrage free. That means there needs to be another analysis in order to be sure that the arbitrage free property is achieved. A potential solution could be to perform the same work, but over a parameterization of a risk free surface. One could try to study how parameters respond to an underlying move, knowing that the output surface will be free of arbitrage by construction.

2 Simulating with the right drift and forward

In the investment banking world, pricing is about taking as much information from the market and use it in order to value non-liquid products. That way, one can believe that the product has been priced according to the market.

For instance, the risk neutral distribution of an underlying at any point T in the future can be known: it is given by vanilla prices. This is for that reason that building a volatility model which fits vanilla prices is possible, we can use the Local Volatility model to do so. Such a model can then be used to price exotic options (even though the price it produces is not perfect due to the unrealistic dynamics this model represents).

In the same manner, forward plays a central role. Indeed, a fundamental risk free relation is:

$$\begin{aligned} F_t^T &= \frac{S_t}{B(t,T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{S_T}{B(T,T)}\middle|F_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^T}\left[S_T|F_t\right] = \mathbb{E}^{\mathbb{Q}^T}\left[S_T^{S_t=x}\right] \end{aligned} \tag{1}$$

Where we successively used:

- Non arbitrage condition on the forward price
- Definition of the forward neutral probability $\mathbb{Q}^T$
- $B(T,T) = 1$ by definition of zero coupon
- Markovian property of EDS solution

Equation (1) tells us that FtT is nothing but the first order moment of S_T. Note that this is consistent with the fact that practitioners see the forward price as the best estimate of the spot at that date. This implies that, when simulating an underlying, one should always make sure that, at any date, the average of the realization is the forward price (which is given by the market).

But how can we achieve this? To answer that question, let us look back at Black and Scholes equation:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{2}$$

$$\Rightarrow S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T} = S_0 e^{rT} e^{-\frac{\sigma^2}{2}T+\sigma W_T} \tag{3}$$

The equation (2) can be splitted in two parts:

A deterministic one (a simple ODE):

$$\frac{dS_t}{S_t} = rdt$$

$$\Rightarrow S_T = S_0 e^{rT} = F_0^T$$

In other words, the forward is the solution to the deterministic part. The forward contains all the information regarding the drift. In particular, it enables us to avoid specifying rates, repo and dividends. They are already in the forward. This is a major point.

A pure stochastic one:

$$\frac{dS_t}{S_t} = \sigma dWt$$

$$\Rightarrow S_T = e^{-\frac{\sigma^2}{2}T+\sigma W_T} = X_T$$

X_T is called an exponential martingale. Note that, as a martingale:

$$E^{\mathbb{Q}^T}[X_T] = E^{\mathbb{Q}^T}[X_0] = E^{\mathbb{Q}^T}\left[e^0\right] = 1$$

A particularity of equation (2) is the fact that its solution is nothing but the product of the solution of the two differential equations:

$$S_T = F_0^T X_T$$

Finally, we notice:

$$E^{\mathbb{Q}^T}[S_T] = F_0^T E^{\mathbb{Q}^T}[X_T] = F_0^T$$

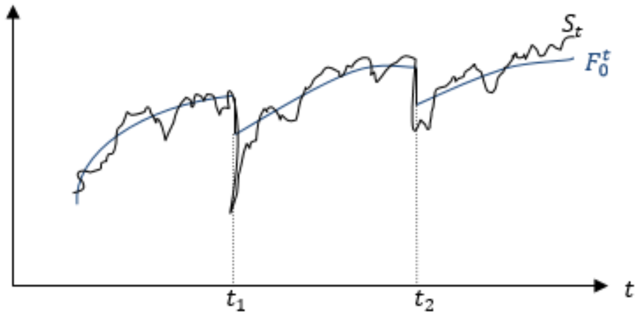
We have just shown that Black & Scholes model meets the fundamental condition given by (1). Now, let us move to any volatility model we want. We know that, under that model, the average should still be the forward. In order to achieve a consistent simulation, one can simply do:

$$S_t = F_0^t X_t \tag{4}$$

Where:

$$\begin{cases} \frac{dX_t}{X_t} & = & \sigma_t dW_t \\ X_0 & = & 1 \end{cases}$$

Proceeding like that will lead to paths where, on average, the spot equals the forward, but the rest of the distribution will depend on the volatility model. The following chart helps us visualize what happens, there is a "forward path", which is the central one and the realization displays an oscillation around it.



Note that, on that chart, t_1 and t_2 represent ex-div dates. In the end, we have just proposed a diffusion scheme where, instead of trying to model rates, repos and dividends, one can simply multiply an exponential martingale by the current forward. Proceeding like that will give the central view of the market. Afterwards, practitioners can focus on modelling the volatility with the level of complexity that suits their needs. They will still be right on average, this is the most important feature.

3 Smoothing in finance

Smoothing is the process by which one tries to lower the impact of a discontinuity or irregular values. In finance, one is particularly interested in smoothing prices and Greeks. There are plenty of methods to reach that goal. Here, we will present the two main techniques used on the market. We shall discuss their importance in practice.

3.1 Price smoothing through payoff

Intuitively, any financial product's price is the price of its replication, which means the price to build another portfolio which behaves the same way as the initial financial product. An exotic product is mainly (but not only) replicated by its underlying(s). We could take a single underlying exotic product as an example. It appears that the quantity of underlying one should detain to replicate is the exotic's delta. Let us note $E()$ the exotic price function (which does depend on the underlying S). By definition:

$$\text{Exotic's\_delta} = \frac{\partial E}{\partial S}$$

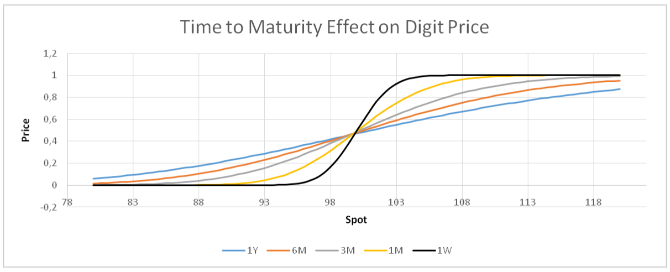
Like a lot of functions, this derivative does depend on S, i.e. the level of the underlying. That means that when the underlying moves, the trader has to adjust his position. This is called dynamic hedging.

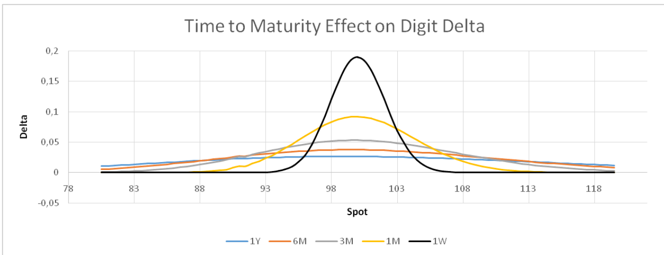
Now that we are aware of that, there is a simple mathematical feature of the payoff that will tell how often a replicating portfolio should be adjusted: its convexity (or concavity). Indeed, the more convex around a point a payoff is, the more the delta increases/decreases, and so the more complicated it is for a trader to manage. On the contrary, the less convex a payoff is, the less the delta moves, and the easier it is for the trader. Smoothing prices through payoff is about changing the payoff to make it less convex. Note that this change must be done in a conservative way (from the investment banking perspective). If the bank sells the product, it should replace the problematic payoff by a less convex but higher one. Of course, if that new payoff is significantly higher than the first one, it will not give a good price and and it is unlikely that the costumer will accept it.

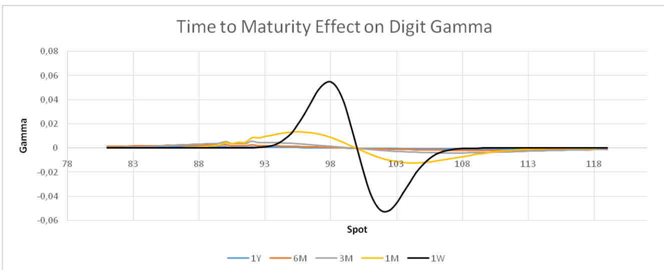
To illustrate that point, let us look at a simple example, a digital option. Its payoff is given by:

$$f(S_T) = 1_{S_T \geq K}$$

Because of the discontinuity around the strike, the price does increase mainly around it. This is particularly the case when Time to maturity or volatility is low. Below are charts of the Time to maturity effect on prices and Greeks of a digital option. It was done under a simple Black & Scholes model, with an annualized volatility equal to 30%, $K = 100$ and $T = 1$:



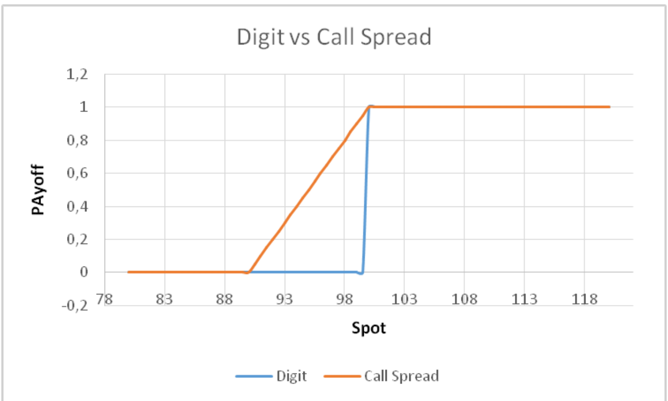




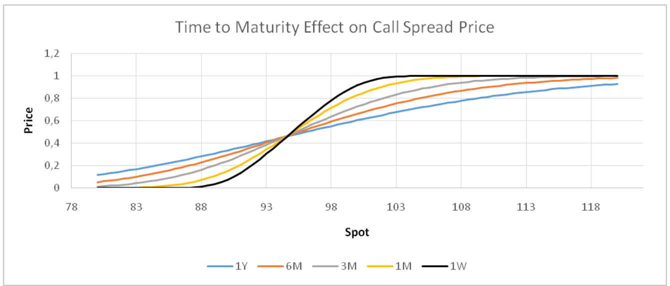
Clearly, one can see that, around the strike, for short-term maturity option, the risk management is not impossible, as the trader is due to buy or sell a large amount of underlying following a price change. The idea here is to replace f by g:

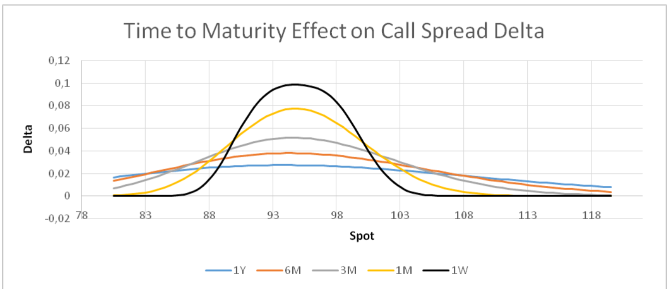
$$g(S_T) = \frac{1}{\epsilon}\left((S_T - (K - \epsilon))_+ - (S_T - K)_+\right)$$

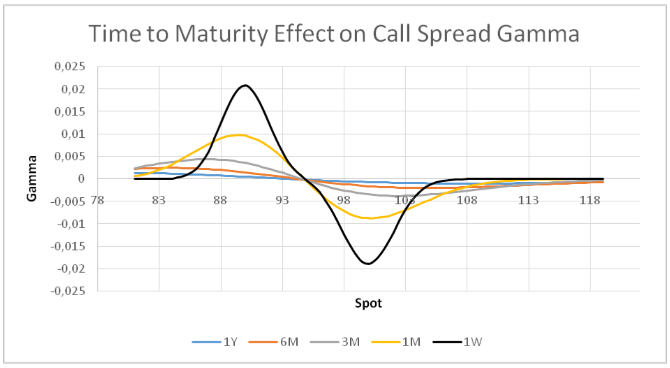
Such a payoff actually has a name: it is a Call Spread. Graphically, using $\epsilon = 10$, one has:



Now, performing the same calculation will lead to the following charts:







We can notice that the shapes are globally the same. Yet, the situation is different, as now the trader will have more time to adjust his delta. Jumps are not as sudden and significant as in the previous case. Having to buy 100M€ of underlying on the market is not the same thing as 70M€. Transaction costs make the first situation (standard digital option) impossible to handle.

Note that over replicating with a linear function is the simplest idea one can have. In practice, other forms of payoff smoothing can include: piecewise function, logistic function...

3.2 Greeks smoothing by calculation

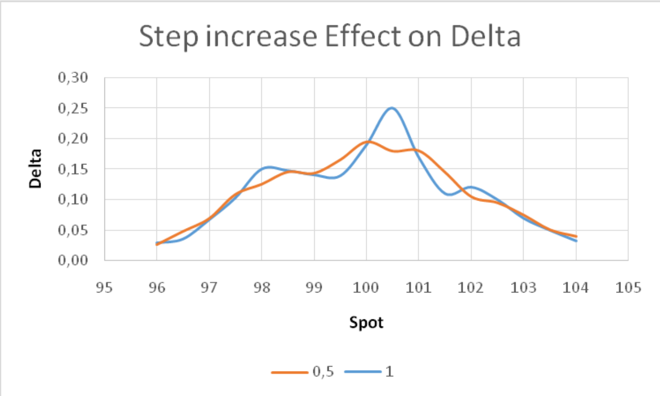
Now that the payoff was modified, what if the trader still finds his Greeks too volatile? Indeed, because exotic prices are computed using Monte Carlo method, prices are not a regular function of the different risk factors, even if the payoff was smoothed. This is mostly due to the noise within the Monte Carlo method.

In practice, one can solve this problem quite easily. Indeed, Greeks are computed with finite difference technique, which simply consists in:

$$\frac{\partial E}{\partial S} = \lim_{h \to 0} \frac{E(S+h) - E(S-h)}{2h} \simeq \frac{E(S+\epsilon) - E(S-\epsilon)}{2\epsilon}$$

In a theoretical world, one should take ϵ as small as possible. But, due to the Monte Carlo noise, this is clearly not something we want. Indeed, taking small ϵ will take into account very small variation which has nothing to do with the underlying variation, but only the numerical method itself. Obviously, traders with experience recognize situation where their deltas (and other Greeks) are not satisfactory. A common technique to get better results is thus to improve the ϵ. By doing so, one will clearly look at the direction of the price curve, i.e. its derivative.

Below is a chart that compares delta with $\epsilon = 0.5$ vs $\epsilon = 1$ on the same digital option as previously, but priced with a Monte Carlo (still Black Scholes model).



Even though the delta is still not perfectly smooth, it is an improvement with regards to a smaller ϵ.

Conclusion

In this paper, we have raised and discussed about three different major topics in equity modelling:

- Implied volatility dynamics problem
- The forward price and how to use it in the modelling process
- Smoothing techniques and their key role

These three topics are capital as they have a major impact on prices and on the bank's business.

We have detailed and shown that using the forward when simulating allows one to get the right average of the underlying at maturity. Even though one uses a wrong volatility modelling, one can be sure to be right on average. This is a major improvement.

Traders manage their position. Their main tools are Greeks, which tell them by how much their portfolio will change when the underlying parameters move. Because of market impact and limits, they cannot afford to buy and sell large quantities at all time. But structured products actually display these features! Smoothing is an essential part of quantitative modelling and fundamental for a bank.

Finally, because of a finite storage capacity, banks have to make choices. Their most crucial choice lies in the implied volatility which is not available at all time. But if one wants to price according to the market and provide prices with no arbitrage, it is important to rely on the best implied volatility dynamics as possible. In this study, we have proposed a new one which seems to be a good step in modelling a realistic dynamics.

A propos d'Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduit à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d'aider nos clients à se concentrer sur l'essentiel, à savoir leur performance.
Pour accomplir cette mission, nous nous appuyons sur trois ingrédients : habileté technique, savoir-faire fonctionnel et innovation.
Ceci au service d'une ambition : dompter la complexité pour simplifier la vie de nos clients.

« *Run the bank* » avec Awalee !

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