Randomized Quasi Monte Carlo

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Summary

Introduction

The 2008 financial crisis has suggested to various financial players to give a greater attention to risk and see how to manage it efficiently. While a small number of banks are prepared for the implied regulatory changes and are actively managing it, the complexity and cost of implementing the necessary infrastructure remain a huge challenge for most financial institutions.

This shows that modeling the risk factors is a fundamental part of the risk management and it introduces changes on the day-to-day pricing and hedging on transactions within this market. This requires an effective and particularly fast method for the numerical process used in pricing.

To overcome this issue, banks have developed a methodology based on the classic Monte Carlo method (MC), which is very costly in terms of computational resources (in terms of time and computational power). These resources grow larger with the number of assets that are analyzed, each risk factor can require several thousands of simulated paths, increasing the complexity of the computational process. As an alternative to the usual computing methods, many banks have decided to tackle the issue of Monte Carlo computing cost by using GPUs instead of the usual architecture using only CPUs. This allows the development of a parallel computing framework which is ideal to estimate the parameters used in the derivatives pricing model.

Quasi-Monte Carlo (QMC) is also widely used because it provides a better convergence rate. Unfortunately, in practice, we see that the method gives less accurate results compared to MC when dimension increases. Furthermore, it does not provide a confidence interval.

In this note, we present Randomized Quasi-Monte Carlo (RQMC), considered as a midway between MC and QMC. The aim of this new methodology of pricing is to improve the rate of convergence of MC and provide a confidence interval.

It is a hybrid method which uses scenarios generated using MC to have a confidence interval and scenarios generated using QMC to get an optimal rate of convergence.

The first two sections will give a formal presentation of MC and QMC in a general framework, that is to say, the variable of which we seek to compute the expectation could be any financial asset or metric such as an option payoff or a Value at risk... Then we will move to the main subject of the article which is RQMC. Finally, we will illustrate the new method by pricing arithmetic Asian options.

1 Monte CARLO

The basic principle of Monte Carlo is to use the Strong Law of Large Numbers: if $(X_k)_{k\geq 1}$ denotes a sequence of independent realizations of an integrable random variable X (i.e. identically distributed random variables) then:

$$\overline{X}_n := \frac{\sum_{k=1}^n X_k}{n} \to m_X = \mathbb{E}[X] \text{ as n} \to \infty \text{ almost surely}$$

The error can be controlled thanks to the Central Limit Theorem which says that: If X is square integrable then:

$$\sqrt{n}(\overline{X}_n - \mathbb{E}[X]) \to N(0, Var(X)) \text{ as n} \to \infty \text{ in distribution}$$

where Var(X) is the variance of X.

Therefore, the confidence interval at level α of the Monte Carlo simulation is given by:

$$I_n = \left[\overline{X}_n - q_\alpha\sqrt{\frac{\overline{V}_n}{n}}, \overline{X}_n + q_\alpha\sqrt{\frac{\overline{V}_n}{n}}\right]$$

where q_α is defined as $\mathbb{P}(m_X \in I_n) \approx \mathbb{P}(N(0,1) \leq q_\alpha) = 1 - \frac{\alpha}{2}$ and $\overline{V}_n$ is the empirical variance of the MC estimator.

The rate of convergence for MC is then $O(\frac{1}{\sqrt{n}})$.

2 Quasi-Monte Carlo

In the rest of the article we will denote $I = [0,1]^d$, d is the dimension of the random variable of interest. Quasi-Monte Carlo (QMC) is a deterministic alternative method to Monte Carlo: the pseudo-random numbers used to generate random variables in MC simulation are replaced by deterministic computable sequences of I valued vectors which may speed up significantly the rate of convergence. Such sequences are called Low discrepancy sequences.

Next, we will define a uniformly distributed sequence and a Low discrepancy sequence.

Definition 1: Uniformly distributed sequence

A sequence $u = (u_n)_{n\geq 1}$ is uniformly distributed on I if for every $x = (x^1, .., x^d) \in I$, its Star discrepancy

$$D_n^*(u) := \sup_{x\in I}\left|\frac{1}{n}\sum_{k=1}^n 1_I(u_k) - \prod_{i=1}^d x^i\right|$$

satisfies $D_n^*(u) \to 0$ as n $\to \infty$

Definition 2: Low discrepancy sequence

A I valued sequence $\epsilon = (\epsilon_n)_{n\geq 1}$ is a sequence with low discrepancy if:

$$D_n^*(\epsilon) = O(\frac{\log(n)^d}{n}) \text{ as n} \to \infty$$

So, when using QMC the rate of convergence is $O(\frac{\log(n)^d}{n})$.

For more details about low discrepancy sequences we refer to [1].

Next, we will define the Koksma-Hlawka inequality that controls the error when using QMC.

Koksma-Hlawka Inequality

Let $\epsilon = (\epsilon_1, .., \epsilon_n)$ be an n-tuple of I valued vectors and let $f : I \to \mathbb{R}$ be a function with finite variation (in the measure sense). Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\epsilon_k) - \mathbb{E}[f(U)] \right| \leq V(f) D_n^*(\epsilon)$$

where U is a uniform random variable and $V(f)$ is the Finite Variation in the measure sense of f.

It's sometimes difficult to use such inequality as $V(f)$ is not straightforward.

But if $V(f)$ is known the previous inequality could be very useful, combined with the attractive rate of convergence, the QMC method can be a strong alternative to classic MC especially in low dimension.

But a major drawback of QMC is that it doesn't provide a confidence interval.

3 Randomized Quasi-Monte Carlo

The idea is to introduce some randomness in the QMC method in order to produce a confidence interval. This approach also leads to a variance reduction compared to classic Monte Carlo. We refer to [2] for details about pricing options using RQMC.

We will introduce two methods of randomization: Shifted low discrepancy sequence and Scrambled sequence methods.

Method 1: Shifted low discrepancy sequence

We consider the following estimator:

$$\overline{X}_{NI} = \frac{1}{NI} \sum_{i=1}^{I} \sum_{k=1}^{N} f(\{X_i + \epsilon_k\})$$

instead of

$$\frac{1}{N} \sum_{k=1}^{N} f(\{\epsilon_k\})$$

in QMC.

The X_i are independent copies of the uniform random variable X generated as in classic MC. $\{.\}$ is the fractional part.

In the following section we will show that the variables $(\{a + \epsilon_k\})$ are effectively uniform. That is to say we will prove that if $(\epsilon_k)_{k\geq 1}$ is a uniformly distributed sequence and $a = (a^1, .., a^d) \in \mathbb{R}^d$, then $(\{a + \epsilon_k\})_{k\geq 1}$ is uniformly distributed.

Demonstration:

We will use the Weyl criterion, see [3] for details. We have by such criterion that if $(\epsilon_k)_{k\geq 1}$ is uniformly distributed:

$$\forall p \in \mathbb{N}^d, p \neq 0_{\mathbb{N}^d} \qquad \frac{1}{N} \sum_{k=1}^{N} e^{2i\pi(p|\epsilon_k)} \to \text{ as N} \to \infty$$

Now,

for $p = (p^1, .., p^d), a = (a^1, .., a^d) \in \mathbb{R}^d$

$$\begin{aligned} \frac{1}{N} \sum_{k=1}^{N} e^{2i\pi(p|\{a+\epsilon_k\})} &= \frac{1}{N} \sum_{k=1}^{N} e^{2i\pi \sum_{i=1}^{d} p^i(\{a^i\}+\{\epsilon_k^i\})} \\ &= e^{2i\pi(p|\{a\})} \frac{1}{N} \sum_{k=1}^{N} e^{2i\pi(\sum_{i=1}^{d} p^i \epsilon_k^i)} \\ &= e^{2i\pi(p|\{a\})} \frac{1}{N} \sum_{k=1}^{N} e^{2i\pi(p|\epsilon_k)} \to 0 \text{ as N} \to \end{aligned}$$

Therefore, the same criterion implies that $(\{a + \epsilon_k\})_{k\geq 1}$ is uniformly distributed.

Method 2: Scrambled sequence

Briefly the idea is to generate:

$$\frac{1}{N} \sum_{k=1}^{N} f(\{\epsilon_{k+T}\})$$

where $(\epsilon_n)_{n\geq 1}$ and is a low discrepancy sequence and T is a discreet random variable defined over $\{1, ..., N\}$. A research activity would be to determine which choice of the distribution of T will reduce the bias of the estimator.

In our article, we chose to use Poisson distribution because it is one of the most important discreet distributions and also for its use in jump processes in finance.

For variance Analysis and confidence interval coverage we will only focus on the first method of randomization.

Variance Analysis

The variance of the estimator is

$$\frac{1}{I} Var\left(\frac{1}{N} \sum_{k=1}^{N} f(\{U + \epsilon_k\})\right) = \frac{\sigma^2}{I}$$

this hybrid method should be compared to regular Monte Carlo of size IN through their respective variances. It is clear that we will observe a variance reduction if and only if

$$\frac{\sigma^2}{I} < \frac{Var(f(U)}{IN}$$

i.e.

$$\sigma^2 < \frac{Var(f(U)}{N}$$

On the other hand:

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\epsilon_k) - \mathbb{E}[f(U)] \right| \leq V(f) D_n^*(\epsilon)$$

Consequently, we can write:

$$\begin{aligned} \sigma^2 &= Var\left(\frac{1}{N} \sum_{k=1}^{N} f(\{U + \epsilon_k\})\right) \\ &= \mathbb{E}\left[\left(\frac{1}{N} \sum_{k=1}^{N} f(\{U + \epsilon_k\}) - \mathbb{E}\left[\frac{1}{N} \sum_{k=1}^{N} f(\{U + \epsilon_k\})\right]\right)^2\right] \\ &= \mathbb{E}\left[\left(\frac{1}{N} \sum_{k=1}^{N} f(\{U + \epsilon_k\}) - \mathbb{E}\left[f(U)\right]\right)^2\right] \\ &\leq \sup_{u\in I} \left|\frac{1}{n} \sum_{k=1}^{N} f(\{u + \epsilon_k\}) - \mathbb{E}\left[f(u)\right]\right|^2 \\ &\leq \sup_{u\in I} V(f_u)^2 D_n^*(u)^2 \end{aligned}$$

where f_u is the function defined as $f_u(v) = f(\{u + v\})$.

Therefore, using the upper bound of the star discrepancy of the sequence $\epsilon = (\epsilon_n)_{n\geq 1}$ there exists some constant C_ϵ such that:

$$\sigma^2 \leq C_\epsilon^2 \frac{log(N)^{2d}}{N^2} \qquad N \geq 1$$

It is clear that randomized QMC provides a very significant variance reduction for the same complexity. But if the dimension d increases one must have in mind that the variance can be high.

4 Confidence interval

We will construct a confidence interval using the Berry–Esseen theorem which specifies the error of approximation between the normal distribution and the true distribution of the empirical mean $\overline{X}_n$. See [4] for more details.

Briefly the theorem states that given that $\mathbb{E}[X] = 0, E[X^2] = \sigma^2 > 0$ and $\mathbb{E}[|X|^3] = \rho < \infty$ If we define F_n as the cumulative distribution function of $\frac{\overline{X}_n\sqrt{n}}{\sigma}$ and Φ the cumulative distribution function of the standard normal distribution, then for all real x and non null integer n, there exists a positive constant C such that:

$$|F_n(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3\sqrt{n}}$$

Application to normal approximation

Consider the randomized quasi-Monte Carlo estimator

$$\overline{X}_{NI} = \frac{1}{NI}\sum_{i=1}^{I}\sum_{k=1}^{N} f(\{X_i + \epsilon_k\})$$

and F_{NI} The cumulative distribution function of $\frac{\overline{X}_{NI}\sqrt{NI}}{\sigma}$ Then

$$|F_{NI}(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3\sqrt{NI}}$$

Define q_α as $\mathbb{P}(N(0,1) \leq q_\alpha) = 1 - \frac{\alpha}{2}$ We have, using the previous inequality:

$$\begin{aligned}
1-\alpha &= \mathbb{P}(-q_\alpha \leq N(0,1) \leq q_\alpha) \\
&= \Phi(q_\alpha) - \Phi(-q_\alpha) \\
&\leq F_{NI}(q_\alpha + \tfrac{C\rho}{\sigma^3\sqrt{NI}}) - F_{NI}(-q_\alpha - \tfrac{C\rho}{\sigma^3\sqrt{NI}}) \\
&= \mathbb{P}(-q_\alpha - \tfrac{C\rho}{\sigma^3\sqrt{NI}} \leq \tfrac{\overline{X}_{NI}\sqrt{NI}}{\sigma} \leq q_\alpha + \tfrac{C\rho}{\sigma^3\sqrt{NI}})
\end{aligned}$$

Therefore, this provides the following confidence interval of level at least α:

$$I_{NI} = \left[\overline{X}_{NI} - q_\alpha\sqrt{\frac{\sigma^2}{N}} - \frac{C\rho}{\sigma^2 I}, \quad \overline{X}_{NI} + q_\alpha\sqrt{\frac{\sigma^2}{N}} + \frac{C\rho}{\sigma^2 I}\right]$$

The size of the interval is bigger than that of classic MC so it's less accurate. But if we dynamically increase N (number of Quasi random simulation) and I (number of random simulation) the term $\frac{C\rho}{\sigma^2 I}$ tends to zero and consequently we can use the classic MC interval:

$$I_n = \left[\overline{X}_n - q_\alpha\sqrt{\frac{V_n}{n}}, \quad \overline{X}_n + q_\alpha\sqrt{\frac{V_n}{n}}\right]$$

But the computation time will increase as we need to estimate another moment and make the product NI go to infinity.

5 Application to option pricing

We will apply the previous results to the pricing of an Asian arithmetic call and Asian arithmetic put. We will consider a geometric Brownian motion (Black-Scholes world) for the underlying stochastic equation.

We will set the volatility to 0.1, the risk-free rate to 0.01, the maturity to 1 year and the current price to 50. The value of the Arithmetic Asian Call is:

$$\mathbb{E}\left[max(\frac{1}{T}\int_0^T S(t)dt - K, 0)e^{-rT}\right]$$

Since we do not have a closed formula for the price of the Arithmetic Asian call, we can use classic MC with a very high number of simulations (100 Millions) and a high number of discretization (100) in the integral calculation to get a benchmark. In practice, we cannot afford to price in such costly conditions. The benchmark results are:

K	CALL Price	PUT Price
45	5.99	0.01
50	1.84	0.62
55	0.17	3.70

where K is the strike of the option.

Testing methodology

The MC estimator is:

$$\frac{\sum_{j=1}^{n} max\left(\frac{\sum_{i=1}^{m} S(\frac{(2i-1)T}{2m})_j}{m} - K, 0\right)}{n}$$

We used the midpoint method to compute the integral. We run a classic MC with 1 million simulations.

In the following, NT is the number of time discretization. We run 1000 QMC simulation using Van der Corput sequence and 1000 MC simulation.

We use two methods to generate a standard normal variable for the randomized Quasi Monte Carlo methods: Box Muller and the Inverse cumulative distribution function.

RQMC-1 refers to the first method of randomization and RQMC-2 refers to the second one.

Numerical results

The results for the pricing of the Asian arithmetic options are:

		Call Price - Box-Muller			Price relative error			Variance			Time taken(s)		
NT	K	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2
16	45	9,64906	6,34674	5,64838	0,61086	0,05956	0,05703	11,6958	4,87593	1,97086	6,498	8,59900	6,87100
	50	4,75843	1,73218	0,94066	1,58610	0,05860	0,48877	10,4158	3,02878	1,19940	5,969	7,76600	7,32300
	55	1,1995	0,019964	0,00000	6,05588	0,88256	1,00000	3,86416	0,01925	0,00000	6,821	8,17400	7,66000
64	45	7,12414	6,18972	5,89044	0,18934	0,03334	0,01662	9,484	1,60596	0,61444	22,985	37,63000	38,18400
	50	2,54672	1,30298	0,91928	0,38409	0,29186	0,50039	6,38079	1,20907	0,52711	21,404	38,31000	35,24100
	55	0,318593	0,00000	0,00000	0,87408	1,00000	1,00000	0.923003	0,00000	0,00000	22,713	38,98500	36,32800

		Call Price - Inverse			Price relative error			Variance			Time taken(s)		
NT	K	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2
16	45	9,64906	6,26287	4,97159	0,61086	0,04555	0,17002	11,6958	2,91534	0,08483	6,498	9,48700	9,91600
	50	4,75843	1,50255	0,10091	1,58610	0,18340	0,94516	10,4158	1,86072	0,04419	5,969	10,14500	10,31800
	55	1,1995	0,004739	0,00000	6,05588	0,97212	1,00000	3,86416	0,00437	0,00000	6,821	10,04500	10,74400
64	45	7,12414	6,18812	4,93356	0,18934	0,03308	0,17637	9,484	1,10168	0,02642	22,985	46,81100	48,28800
	50	2,54672	1,26849	0,05012	0,38409	0,31060	0,97276	6,38079	0,82514	0,01118	21,404	47,98400	48,61100
	55	0,318593	0,00000	0,00000	0,87408	1,00000	1,00000	0.923003	0,00000	0,00000	22,713	49,60500	47,86200

		Put Price - Box-Muller			Price relative error			Variance			Time taken(s)		
NT	K	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2
16	45	0,000661	0	0,00000	0,93395	1,00000	1,00000	0,000879	0,00000	0,00000	6,198	8,89500	8,68000
	50	0,110031	0,385435	0,29228	0,82253	0,37833	0,52857	0,24653	0,51187	0,22158	5,632	8,78900	6,90500
	55	1,5511	3,67322	4,35162	0,58078	0,00724	0,17611	4,1242	4,71002	1,97086	6,394	8,25100	6,95700
64	45	0,005247	0	0,00000	0,47525	1,00000	1,00000	0,00781	0,00000	0,00000	21,452	38,05500	32,52800
	50	0,427826	0,113259	0,02885	0,30996	0,81732	0,95347	1,00668	0,10174	0,03430	21,154	38,76800	31,96900
	55	3,1997	3,81028	4,10956	0,13522	0,02981	0,11069	6,60478	1,60596	0,61444	20,401	38,05400	32,35800

		Put Price - Inverse			Price relative error			Variance			Time taken(s)		
NT	K	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2	MC	RQMC-1	RQMC-2
16	45	0,000661	0	0,00000	0,93395	1,00000	1,00000	0,000879	0,00000	0,00000	6,198	10,04100	8,87800
	50	0,110031	0,239684	0,12932	0,82253	0,61341	0,79142	0,24653	0,33434	0,01454	5,632	10,25700	9,35100
	55	1,5511	3,74187	5,02841	0,58078	0,01132	0,35903	4,1242	2,87551	0,08483	6,394	9,89300	8,80300
64	45	0,005247	0	0,00000	0,47525	1,00000	1,00000	0,00781	0,00000	0,00000	21,452	48,00800	44,53600
	50	0,427826	0,080369	0,11656	0,30996	0,87037	0,81200	1,00668	0,07265	0,00355	21,154	50,80400	44,63600
	55	3,1997	3,81188	5,06644	0,13522	0,03024	0,36931	6,60478	1,10168	0,02642	20,401	47,38300	43,75000

Remarks

RQMC-1:

We notice that whether it's a CALL or PUT the variance is reduced for this method relatively to classic MC. One striking thing is that in all the cases, the inverse method reduces the variance more than Box-Muller method.

In terms of accuracy for the CALL Box muller is more accurate for a 16 discretization whereas the inverse method becomes a little more accurate for the 64 discretization.

So, we conclude that the smaller values the more accurate the method. For the put we cannot say which method of generation (inverse or Box Muller) is the best: Depending on the strike one method outperforms the other and vice-versa.

As a result, this method gives prices that are close to the previous method although it outperforms it in 3 cases out of 6 for the call and 5 cases out of 6 for the PUT when Box muller is used.

RQMC-2:

We notice the for the CALL the first method of randomization gives more accurate results in all the 6 cases.

Concerning the PUT, it's better to use the second method of randomization for out-of-the money and in-the-money options and use the first method of randomization for in-the-money options to get the best relative accuracy.

Concerning computation time, the second method is a little faster and it reduces the variance more than the previous one.

6 Perspectives

In this section we expose some enhancements we could do to compare the various method or to make them better.

For the first Randomized Quasi Monte Carlo Method no choice of the partition between the number of Quasi random simulations (N) and random simulations (I) we could try, for a given number of simulation A, different values of N and I satisfying the condition NI=A. We could then find the optimal partition. For the second Randomized Quasi Monte Carlo Method, we only tried the "Poisson" distribution. We could try different discrete distribution and see the optimal one in terms of the size of the confidence interval, accuracy, variance reduction and computation cost.

As a conclusion, depending on the function of which we want to compute the expectation, the performances of the various methods vary.

We need to study the payoff function and make a trade-off between the criterions (accuracy, variance reduction, computation time and confidence interval) to choose one of the methods.

Conclusion

In this paper, we have presented the Randomized Quasi-Monte Carlo and thus show that it is possible to innovate and propose new modeling methods to facilitate the pricing of derivatives.

We first presented classic Monte Carlo and Quasi Monte Carlo. Then the new RQMC has been illustrated in the case of the Asian arithmetic option using a Geometric Brownian Motion for the underlying which is one of the most fundamental processes in quantitative finance. We have also proposed two methods of randomization.

The proposed method overcomes the drawback of QMC by providing a confidence interval without compromising variance reduction and the fast convergence rates given by low discrepancy sequences.

Nevertheless, it remains to be seen how we can effectively choose the partition of random and deterministic generation so that the size of the confidence interval becomes smaller and the rate of convergence remains fast.

References

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A propos d'Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduits à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d'aider nos clients à se concentrer sur l'essentiel, à savoir leur performance.

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