The Multi-Curve Framework: A Practitioner's Guide

Study carried out by the Quantitative Practice
Special thanks to Romuald OSSENI

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Summary

Introduction

For several years, practitioners have build a single curve to discount and forecast cash flow within interest rate derivatives market. In fact, using the same curve for discounting and forwarding cash flows was not an "aberation" as even if numbers of transactions in the interest rate market were not risk free, the credit and liquidity risk were negligible upon the 2007-2008 crisis, then were found to cause an important impact on the interest rate derivative prices after the crisis. For example, prior to the 2007-2008 financial crisis, OIS and Euribor rates were found to be similar. After that crisis, the spread between these curves (Euribor Vs OIS) reached 222 bps in july 2008, then decreased, but never went back to its pre-crisis level. This main change introduced by the crisis has led the practitioners to update the curves construction technics. The modern multi-curve framework has been introduced to take this change into consideration.
In this paper, we acknowledge the changes rational from single to multi-curve in section 1, while section 2 formalizes the instruments used to retrieve the forward curves within the multi-curve framework, with a focus to the Euro market. However, the listed considerations would be valid whether we considered other markets.

1 The Rational of Switching From Single to Multi-Curve

In this section, we focus on the EUR market where three majors changes has occured since the 2007-2008 financial crisis. These changes have significant impact on the prices of the interest rate derivatives instruments and are at the base of switching from single to multi-curve:

- The large spread observed between the OIS and Euribor curves,
- The breaking down of the well known relation between FRA rates and forward rates implied by two consecutives deposits,
- The explosion of the basis swap spread.

We are now going to highlight the consequence of these changes within the interest rate derivatives market in the post crisis.

1.1 Euribor and OIS Spread

The first reason of significant widening in Euribor and OIS spread (see figure 1) is a consequence of the difference in the credit and liquidity risk between Euribor and Eonia rates [1]. In fact, before the crisis, banks were willing to lend to each other because a bankruptcy among huge financial institutions was deemed to be very unlikely, especially in a short period of time, typically for 3 or 6 months. After Lehnman Brothers bankruptcy, some banks started having less confidence in proposing loans to each other as some of them became less creditworthy, which has led them to increase their loan interest rates amongst which, the Euribor. In summary, there is a greater liquidity and credit risk associated with the Euribor such that the banks, during and after the crisis were viewed as risky in the sense that they were either illiquid or insolvent. So the Euribor rate increased to reflect that risk premium.

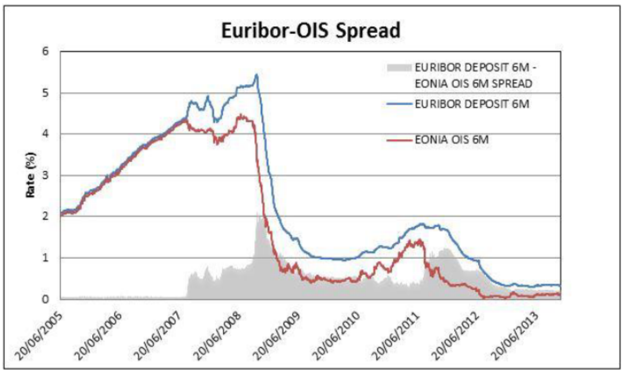

Figure 1: Divergence between Euribor 6M and OIS 6M deposit rates between 20^{th} June 2005 and 18^{th} November 2013 of time intervale. It appears that both OIS and Euribor chases each other until august 2007, then suddenly start diverging and never back to level before the crisis, each one incorporating its own credit and liquidity risk. The corresponding spread is shown in grey. Source: [1]

The second reason of spread explosion is the segmentation of the interest rate market within sub-areas corresponding to interest rates of different tenors. In fact, until the crisis, the OIS 6M and the Euribor 6M spread was negligible because 6M deposit was equivalent to two consecutives 3M deposit in term of risk, so that no risk premium was required for the first or second lending strategy. In other words, the financial world was not tenor-dependent because all the institutions participating in the interbank market were considered unlikely to default, regardless the duration of the lending. So, after the crisis, the observed value of spread (see figure 1) didn't allow using a unique curve indifferently for all tenors as a correct practice.

1.2 Breaking-Down of the Equality Relation between FRA and Implied Forward Rates

A FRA is a forward contract that allows the buyer to lock in the contract rate to be paid at a future date while implied forward rate is retrieved from replication [2]. In fact, through a no-arbitrage argument, the implied forward rate $F\left(t, T_{j-1}, T_j\right)$ is the fixed rate corresponding to the period $\left[T_{j-1}, T_j\right]$. Within the single curve framework where a same curve was used for discounting and funding, the implied forward rate was closed to the FRA so that the equality below was valid:

$$FRA\left(t, T_{j-1}, T_j\right) = F\left(t, T_{j-1}, T_j\right) = \frac{1}{T_j - T_{j-1}}\left[\frac{P\left(t, T_{j-1}\right)}{P\left(t, T_j\right)} - 1\right] \tag{1}$$

where t, T_{j-1} and T_j is respectively the settlement, reset and maturity date.
The relation states that the forward value at time t can be replicated by a linear combination of zero-coupon bonds, one maturing at reset date T_{j-1} and another maturing at settlement date T_j. Hence, this result asserts that all participants have equal utility between borrowing for two three-month periods and borrowing for a single six-month period. In [1], the authors have compared the Euribor FRA quoted and the corresponding Euribor FRA (Euribor implied forward rate) retrieved from the replication (equation 1) under tenors from 1M to 12M. The results (see figure 2) show that, on 30^{th} Decembre 2011, the Euribor FRA quoted rate was always smaller than the corresponding Euribor FRA rate implied by replication. Based on that observation, it becomes evident that the single curve framework is invalid so that one needs to switch to the multi-curve framework.

Euribor FRA Replication (30 Dec. 2011)

Euribor Depo. Maturity	Euribor Deposit Quote (%)	Euribor FRA	Euribor FRA Quote (mid, %)	Euribor FRA Replica (%)	Difference Replica-Quote (bps)
1M	0.980	1Mx4M	1.223	1.500	27.7
2M	1.150	2Mx5M	1.130	1.677	54.7
3M	1.310	3Mx6M	1.067	1.804	73.7
4M	1.380	4Mx7M	1.016	1.948	93.2
5M	1.460	5Mx8M	0.964	2.080	111.6
6M	1.560	6Mx9M	0.931	2.103	117.2
7M	1.620	1Mx7M	1.471	1.728	25.7
8M	1.690	2Mx8M	1.365	1.883	51.8
9M	1.740	3Mx9M	1.292	1.958	66.6
10M	1.790	4Mx10M	1.246	2.073	82.7
11M	1.840	5Mx11M	1.200	2.154	95.4
12M	1.900	6Mx12M	1.172	2.243	107.1
18M	1.860	12Mx18M	1.125	1.736	61.1
24M	1.870	18Mx24M	1.224	1.868	64.4
		12Mx24M	1.481	1.800	31.9

Figure 2: Comparison between the Euribor FRA and the corresponding Euribor Forward rates (Euribor FRA replicas). Source [1]

1.3 Increasing Basis Swap Spread

The third and last evidence is the emergence of significant basis swap spreads. In fact, the basis swap is similar to a vanilla swap but both legs are floating although with different tenors. As pointed out by Bianchetti and al. in [1], the basis swap with the same maturity follows each other so closely that there were barely no spread.

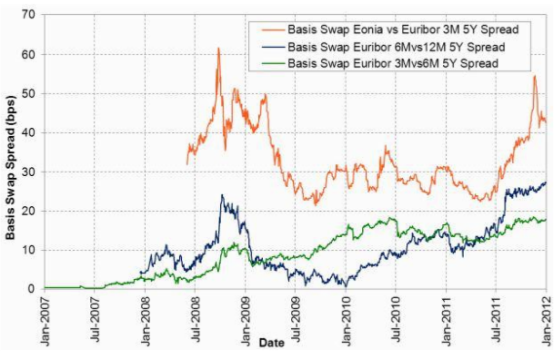

Figure 3: Euribor basis spread historical market data from 2007 to 2012. Note that regardless to the tenor, the basis spread is closed to zero prior to july 2007, then a growing trend starting from august 2007. Source [1]

After the crisis, a spread appeared and showed a growing trend. That evidence of spread highlights to practitioners how the market evaluates the two different tenors within the basis swap.
Now that the majors consequences of the financial crisis within the market rate have been reviewed, we are going to introduce the multi-curve framework and the changes occured to the interest rate derivatives pricing.

2 The Multi-Curve Practitioner's Guide

One of the post-crisis effect was that Over-The-counter contracts became heavily collaterised and this trend was reinforced by current regulatory such as European Market Infrastructure Regulation in Europe (EMIR) and Dodd-Frank in the USA. The common collateral agreement was the *Credit Support Annex*(*CSA*) of the ISDA [3, 4, 5, 6] agreement.
In this section, we assumed that we are perfect CSA collateral agreement, so that the price of each contract is given by:

$$V(t) = \sum_{j=1}^{n} P\left(t, T_j\right) \mathbb{E}_t^{Q^{T_j}}\left[C_j\right] \tag{2a}$$

$$P\left(t, T_j\right) = \mathbb{E}_t\left[exp\left(-\int_t^{T_j} r(u)\, du\right)\right]$$

where Q^{T_j} is the associated measure, C_j is the cash flow for times T_j, which may depend on the spot libor rate $L_x\left(T_{j-1}, T_j\right)$ with $x = \tau\left(T_{j-1}, T_j\right)$ is the interest rate derivative tenor, $P\left(t, T_j\right)$ the discount factor and $r(t)$ the short rate. So pricing instrument depends on being able to compute the Libor rate and discount factor. In other words, to price an interest rate derivatives with tenor x, we need two curves: a discount curve and a forward curve with tenor x. Understanding that, one can define the multi-curve as the purpose of constructing many curves as the underlying rates tenors are and another curve to discount the cash flow.

2.1 The Discount Curve

In supplement of market segmentation implied by the new conception of the credit and liquidity risk in the interest rate market, another implication is that the shorter is the tenor of a stream of payments in a contract, the lower is the embedded risk. Besides, the shortest tenor available within the market is the overnight rate, which under European market is Eonia. For that reason, the best proxy for the riskless rate is the Eonia [7, 8, 9].
OIS agreements are regular swap agreements in which the floating leg references the overnight rate. Let **T** and **S** represent the floating and fixed leg respectively, such that:

$$\mathbf{T} = \left\{T_0, T_1, \ldots\ldots\ldots, T_n\right\} \text{ Floating leg schedule} \tag{3a}$$

$$\mathbf{S} = \left\{S_0, S_1, \ldots\ldots\ldots, S_n\right\} \text{ Fixed leg schedule} \tag{3b}$$

with conditions: $T_0 = S_0$, $T_n = S_n$

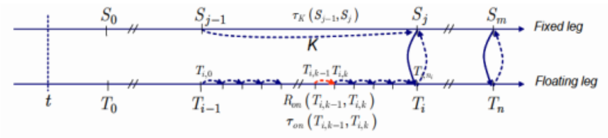

Figure 4: Illustration of the cash flows within the OIS agreement [10]

In figure 4, the fixed leg and the floating leg can have a different frequency, and therefore both require different summation indices.
The value, at time t of the fixed leg paying K at each time S_j is given by:

$$V_{fix}(t) = NK \sum_{j=1}^{m} P\left(t, S_j\right) \tau\left(S_{j-1}, S_j\right) \tag{4}$$

The floating leg is given by:

$$\begin{aligned} V_{flt}(t) &= N \sum_{i=1}^{n} P(t, T_i)\, \mathbb{E}_t^{Q^{T_i}}\left[R_{on}(T_i, T_i)\right] \tau(T_{i-1}, T_i) \\ &= N \sum_{i=1}^{n} P(t, T_i)\, R_{on}(t, T_i)\, \tau(T_{i-1}, T_i) \\ &= N\left[P(t, T_0) - P(t, T_n)\right] \end{aligned}$$

where:

$$\mathbb{E}_t^{Q^{T_j}}\left[R_{on}(T_i, T_i)\right] = R_{on}(t, T_i) = \frac{1}{\tau(T_{i-1}, T_i)}\left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right]$$

The fair OIS rate is the rate, K, which makes $V_{fix} = V_{flt}$, to give:

$$K = \frac{P(t, T_0) - P(t, T_n)}{\sum_{j=1}^{m} P\left(t, S_j\right) \tau\left(S_{j-1}, S_j\right)} \tag{5}$$

The OIS discount curve term structures can be extracted by the above formula as:

$$P(t, T_n) = \frac{P(t, T_0) - K \sum_{j=1}^{m-1} P\left(t, S_j\right) \tau\left(S_{j-1}, S_j\right)}{1 + K\tau(S_{m-1}, S_m)} \tag{6}$$

2.2 The Forward Curve

In the multi-curve framework, there are many forward yield curves as the underlying tenor. In other words, the choice of instruments used in the construction of forward curves is tenor-dependent. Another feature to keep in mind is the liquidity of the selected instruments. In fact, liquidity is a crucial characteristic when determining the subset of all available instruments, as instruments chosen should move coherently as market expectations changes. Having said that, the most common instruments used to retrieve the forward yield curves are:

- The Deposit used to build the overnight (ON) and tomorrow-next (TN) forward term structure,
- The Futures & FRA: these products are used to build the mid-term forward curves, typicall up to **2Y**,
- The Swaps used to build the long term forward curves, typically from one or two to **50Y**,
- The basis swaps, quoted in basis points with maturities ranging from one to **50Y**.

We are now going to briefly describe each one and way to retrieve the forward rate curve from them.

2.2.1 The Deposit

A deposit is a zero coupon contract where a counterparty A (lender) lends a nominal N at T_0 to a counterparty B (borrower), which at maturity T, pays the notional amount back to the lender plus an interest accrued over the period $[T_0, T]$ at a simply compounded rate $R_x(T_0, T)$ previously fixed. Finally, the deposit rate (of tenor $x = T - T_0$) for the period $[T_0, T]$ is given by:

$$R_x(T_0, T) = \frac{1}{\tau(T_0, T)}\left[\frac{1}{P(T_0, T)} - 1\right] \tag{7}$$

2.2.2 The Futures

Interest rate futures are standardized derivatives subject to daily marking to market, making these contracts's credit risk negligible. The market quotes the futures in terms of price rather than in term of rates. The quoted futures price for settlement day T_i is:

$$P_x^{Fut}(t, T_{i-1}, T_i) = 100 - R_x^{Fut}(t, T_{i-1}, T_i) \tag{8}$$

where $R_x^{Fut}(t, T_{i-1}, T_i)$ is the future rate for period $[T_{i-1}; T_i]$ prevailing at t. The corresponding forward rate $F_x(t, T_{i-1}, T_i)$ can be obtained by adding a convexity adjustment so that:

$$F_x(t, T_{i-1}, T_i) = R_x^{Fut}(t, T_{i-1}, T_i) - C_x^{Fut}(t, T_{i-1}, T_i) \tag{9}$$

The $F_x(t, T_{i-1}, T_i)$ is the rate to be used for the construction of the short medium forward curve.

2.2.3 Forward Rate Agreement

The Forward Rate Agreements are quoted on the market, so we can simply take these rates and use them for our forward curve.

2.2.4 The Interest Rates Swaps

The interest rate swaps are financial contracts where one party exchanges a set of floating payments against a set of fixed payments. Consider a payer swap (i.e fixed is paid), the t-value of the swap is:

$$V(t) = V_{flt}(t) - V_{fix}(t) \tag{10}$$

Using the equation 2a and notations 3a, 3b, the time t-value of the fixed leg is given by:

$$\begin{aligned} V_{fix}(t) &= \sum_{j=1}^{m} P\left(t, S_j\right) \mathbb{E}_t^{Q^{S_j}}\left[NK\tau\left(S_{j-1}, S_j\right)\right] \\ &= NK \sum_{j=1}^{m} P\left(t, S_j\right) \tau\left(S_{j-1}, S_j\right) \end{aligned} \tag{11}$$

The time-t value of the floating leg is given by:

$$V_{flt}(t) = N\sum_{i=1}^{n} P(t,T_i)\,\mathbb{E}_t^{Q^{T_i}}\left[L_x(T_{i-1},T_i)\right]\tau(T_{i-1},T_i)$$
$$= N\sum_{i=1}^{n} P(t,T_i)\,L_x(t,T_{i-1},T_i)\,\tau(T_{i-1},T_i)$$

Therefore, the fair swap at inception gives the fair rate K:

$$K = \frac{\sum_{i=1}^{n} P(t,T_i)\,L_x(t,T_{i-1},T_i)\,\tau(T_{i-1},T_i)}{\sum_{j=1}^{m} P\left(t,S_j\right)\tau\left(S_{j-1},S_j\right)} \tag{12}$$

In order to built the forward rate curve, one need to rearrange equation 12 so that, the forward rate term structure is given by:

$$L_x(t,T_{n-1},T_n)\,P(t,T_n)\,\tau(T_{n-1},T_n) =$$
$$K\sum_{j=1}^{m} P\left(t,S_j\right)\tau\left(S_{j-1},S_j\right) -$$
$$\sum_{i=1}^{n-1} P(t,T_i)L_x(t,T_{i-1},T_i)\,\tau(T_{i-1},T_i) \tag{13}$$

The above equation represents a bootstrapping equation from which the forward rate can be computed recursively.

2.2.5 The Basis Swaps

The basis swap is used within the process when the fixed leg and floating leg have a different frequency. In fact, the basis swaps are quoted in the market in terms of the difference between the par rate of the higher frequency leg and the par rate of the lower frequency leg. Basically, the quotation of a basis swap 3M versus 6M is:

$$\Delta_{3M,6M}(t,T_i) = K_{6M}(t,T_i) - K_{3M}(t,T_i)$$

where $K_{6M}(t,T_i)$ and $K_{3M}(t,T_i)$ are the par rate of the swap on 6M Euribor and 3M Euribor respectively.
Pricing a basis swap follows the same reasoning as vanilla swap, replacing the fixed leg in 11 with a second floating with a different tenor:

$$V(t) = N\sum_{j=1}^{m} P\left(t,S_j\right)\left[L_x\left(t,S_{j-1},S_j\right)+\Delta\right]\tau\left(S_{j-1},S_j\right) -$$
$$- N\sum_{i=1}^{n} P(t,T_i)\,L_{x'}(t,T_{i-1},T_i)\,\tau'(T_{i-1},T_i) \tag{14}$$

where $L_x\left(t,S_{j-1},S_j\right)$ is the FRA short frequency leg while $L_{x'}(t,T_{i-1},T_i)$ is the higher frequency and a spread Δ is added to the short-tenor leg.

Conclusion

The multi-curve framework is not significantly different from the single framework, even if the first framework is tenor dependent, unlike the latter one. In other words, in a multi-curve framework, there is one discount factor curve and a forward curve for each tenor. Thus, in the multi-curve, the discount factor is first constructed and used within the forward curves of various tenors computations. The discount factor must be risk-free and constructed from the OIS rate, which is considered as the best proxy to a risk-free rate because many (73.7% of transactions according to ISDA Margin survey) transactions are perfectly collateralized. However, for uncollateralized OTC derivatives, the two parties are exposed to the counterparty risk default. For that, Hull and al. [7] proposed to take into consideration the Credit Valuation Adjustment (CVA) and the Debt Valuation Adjustment (DVA) so that the value ($V_{Uncol}(t)$) of an uncollateralized derivative is given by:

$$V_{Uncol}(t) = V(t) + CVA + DVA$$

where $V(t)$ is the no-default value of derivatives.
Finally, the OIS rate choice (EONIA) is changing due to curve manipulation within the market. For that purpose, follow-up paper will focus on European Short Term Rate (ESTER).

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A propos d'Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

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